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K_0 and Noetherian Group Rings

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The authors use K-theoretic methods to prove that if F is a field of char 0 and G is a torsion free polycyclic-by-finite-group then $F[G]$ is a domain.

One of the most famous and least understood problems about abstract group rings is the zero divisor conjecture: If F is a field and G is a torsion free group, is the group algebra $F[G]$ always a domain?

When G is abelian the conjecture is a variant of the well-known theorem that a polynomial ring is a domain. One may regard the usual argument (based on the additivity of the degree function) as exploiting the fact that G has an ordering compatible with the group operation. Free groups and groups having a normal series with torsion free abelian factors (e.g., torsion free nilpotent groups) can be sufficiently ordered to similarly prove the conjecture in these cases (see [11, p. 95]). A more intricate degree argument shows that the conjecture holds for certain free products with amalgamation [3, 10]. Difficulties arise when one takes a group G for which $F[G]$ is a domain and asks whether the conjecture is still true for a torsion free finite extension of G. The important class of polycyclic groups falls under these considerations; recall that such groups are polyinfinite cyclic extended by finite. Formanek [7] made the first progress in this area by settling the conjecture for supersolvable groups. In a recent remarkable paper [1], K. Brown shows that $F[G]$ is a domain when char $F = 0$ and G is torsion free abelian-by-finite. His techniques inspired our

MAIN THEOREM. *If G is a torsion free, polycyclic-by-finite group and* char $F = 0$ *then $F[G]$ has no zero divisors.*

The first idea for the proof of this theorem came from reworking Brown's use of a result due to Walker [16]. In terms of this paper, it says that there is a rank (or "Euler characteristic") on the Grothendieck group of a noetherian ring which measures how far the ring is from a domain. The second idea is that there are other ranks on a ring, for instance, those induced by traces.

1. GOLDIE'S RANK

In this section we explain the connection between K-theory and the zero divisor conjecture.

G will always denote a group and F will denote a field. R will be an arbitrary ring. If M is a finitely generated (projective) left R-module then M will also denote its class in the Grothendieck group $G_0(R)$ (respectively, $K_0(R)$). If H is a subgroup of finite index in G then $F[G]$ is a finitely generated free left $F[H]$-module. Thus there is a transfer map from $K_0(F[G])$ into $K_0(F[H])$ which we will write $P \mapsto P_H$. The same notation will be used for the corresponding map on G_0's.

Recall that R is (left) regular if it is noetherian and every finitely generated R-module has a finite projective resolution.

LEMMA 1 ("Syzygy" theorem). *If G is a torsion free polycyclic-by-finite group then $F[G]$ is regular.*

Proof. Hall [9] proved that $F[G]$ is noetherian.

It is well known (see [12, p. 65]) that G has a normal subgroup H of finite index which possesses a finite normal series all of whose factors are infinite cyclic. In short, H is poly-Z. In [13, Proposition 6], Serre shows that $cd_F H < \infty$; in Theorem 1 he proves that $cd_F G < \infty$. If $\mathscr{P}$ is a projective $F[G]$-resolution of F and M is any $F[G]$-module then $M \otimes_F \mathscr{P}$ is a projective resolution of M under the diagonal action. Thus $F[G]$ has finite global dimension. ∎

In [8], Goldie defines what he calls a reduced rank for R-modules. His Theorem 1.22 really says that this rank is an integer-valued function which factors through G_0; we write rk_R: $G_0(R) \to Z$. Such functions deserve a name.

DEFINITION. A *G_0-character* (*K_0-character*) on R is a nonzero homomorphism from $G_0(R)$ (respectively, $K_0(R)$) into Z.

We briefly review Goldie's theory of noetherian rings from the point of view of rk. (More details can be found in [8].) If R is prime and noetherian, it has a classical ring of quotients which is isomorphic to the ring of $n \times n$ matrices over some division ring. The n here is $rk_R R$. Thus R is a domain if and only if $rk_R R = 1$. More precisely, the intersection of a minimal left ideal of the matrix ring with R has rk 1.

LEMMA 2. *Suppose $H \lhd G$ and $|G:H| < \infty$. If $0 \neq M$ is an $F[G]$-module then $rk_{F[G]}M > 0$ implies $rk_{F[H]}(M)_H > 0$.*

Proof. $rk_H(M)_H = 0$ means that the $F[H]$-singular submodule of M_H is essential in M_H. That is, if $0 \neq X$ is a left $F[H]$-submodule of M_H then there is a $0 \neq x \in X$ such that $\mathrm{ann}_{F[H]}x$ is essential in $F[H]$. Let $g_1, \ldots, g_n$ be coset representatives for H in G. Then $g_i \,\mathrm{ann}_H x$ is a left $F[H]$-submodule of $F[G] \cong g_1F[H] \oplus \cdots \oplus g_nF[H]$. (We are using $H \lhd G$.) By [8, 1.03], $\sum g_i \,\mathrm{ann}_H x$ is essential in $F[G]$ as a left $F[H]$-module. Thus $F[G]\,\mathrm{ann}_H x$ is an essential left ideal of $F[G]$. But $F[G]\,\mathrm{ann}_H x \subseteq \mathrm{ann}_{F[G]}x$. Therefore, every nonzero left $F[G]$-submodule of M meets the $F[G]$-singular submodule of M' ∎

THEOREM 1. *Let G be a torsion free group with a normal, poly-Z subgroup H of finite index. If there exists a K_0-charcater χ on $F[H]$ such that $\chi(F[H]) = 1$ and $|G:H| \mid \chi(P_H)$ for $P \in K_0(F[G])$ then $F[G]$ is a domain.*

Proof. Notice that χ extends to a G_0-character on $F[H]$ such that $|G:H| \mid \chi(A_H)$ for $A \in G_0(F[G])$. Indeed, if $M \in G_0(F[H])$ then $\chi(M)$ can be calculated from any finitely generated projective resolution of M. (One exists by Lemma 1.) More generally, $K_0(R) = G_0(R)$ for regular rings.

We claim that $\chi = rk_{F[H]}$ as functions on $G_0(F[H])$, or equivalently on $K_0(F[H])$. For the twisted Grothendieck theorem of [5] shows that $K_0(F[H]) = \langle F[H] \rangle$. Thus the two functions agree if they coincide on $F[H] \in K_0(F[H])$. But $F[H]$ is a noetherian domain; as remarked above, $rk_{F[H]}F[H] = 1 = \chi(F[H])$.

We finish by using a trick of Walker [16]. $F[G]$ is a prime ring by [4, Theorem 1]. If it is not a domain we can find a left ideal I such that

$$0 < rk_{F[G]}I < rk_{F[G]}F[G].$$

By Lemma 2,

$$0 < rk_{F[H]}I_H < rk_{F[H]}(F[G])_H .$$

Clearly, $rk_{F[H]}(F[G])_H = |G:H|$. However, $rk_{F[H]}I_H$ cannot be an integer and still be divisible by $|G:H|$. ∎

2. DOMAINS

Theorem 1 is ultimately misleading because the twisted Grothendieck theorem says there is only one possible K_0-character on $F[H]$ which takes the value 1 on $F[H]$. Of course one always has the principal character, π:

$$\pi(P) = \dim_F(F \otimes_{F[H]} P) \qquad \text{for} \quad P \in K_0(F[H]).$$

(Experts will notice that this induces the "geometric" Euler characteristic

$$\pi(B) = \sum (-1)^i \dim_F \mathrm{Tor}_i^{F[H]}(F, B) \qquad \text{for} \quad B \in G_0(F[H]).)$$

The ideas that follow are borrowed from [2].

THEOREM 2. *Let F be a field of characteristic $p > 0$. Suppose G is a torsion free group and H is a normal poly-Z subgroup with $|G : H| = p^n$. Then $F[G]$ is a domain.*

Proof. Let $P \in K_0(F[G])$. $F \otimes_{F[H]} P$ is a finitely generated projective $F[G/H]$-module under the obvious action. In fact, it is a free $F[G/H]$-module since $F[G/H]$ is a local ring. Thus $|G : H| \mid \dim_F(F \otimes_{F[H]} P_H)$. ∎

THEOREM 3. *Let F be a field of characteristic 0. If G is a torsion free polycyclic-by-finite group then $F[G]$ is a domain.*

We limit ourselves to an outline of this theorem, along the lines of Theorem 2. In the next section we provide a second, more detailed proof which does not depend on Swan's deep result about induced representations.

Proof. A specialization argument allows one to assume that F is a number field. Let S denote the ring of algebraic integers in F. S is a (regular) noetherian Dedekind domain with finite class group. Thus if H is poly-Z,

$$K_0(S[H]) = G_0(S[H]) \cong \langle S[H] \rangle \oplus \text{a finite group}.$$

Choose H as in Lemma 1. Since characters kill torsion in G_0, the argument of Theorem 1 proves that $S[G]$ (and hence $F[G]$) is a domain provided

$$|G : H| \mid \mathrm{rank}_S(S \otimes_{S[H]} P) \qquad \text{for} \quad P \in K_0(S[G]).$$

Notice that $\mathrm{rank}_S(S \otimes_{S[H]} P) = \dim_F(F \otimes_S Q)$, where $Q = S \otimes_{S[H]} P$ is a finitely generated projective $S[G/H]$-module. Swan's theorem [15] states that if S is a ring of algebraic integers, Y is a finite group and Q is a finitely generated projective $S[Y]$-module then $F \otimes_S Q$ is a free $F[Y]$-module. Thus $|G : H| \mid \dim_F(F \otimes_S Q)$. ∎

3. THE STALLINGS–FORMANEK CHARACTER

If R is a ring then $M_n(R)$ denotes the ring of $n \times n$ matrices with entries in R. If $m \in M_n(R)$ then tr m denotes the sum in R of the diagonal entries of m.

Assume $R^a \simeq R^b$ implies $a = b$. Stallings [14] defines a trace map T on R

to be an additive function into an abelian group satisfying $T(rs) = T(sr)$ for $r, s \in R$. T extends to a trace, $\tilde{T}$, on $M_n(R)$ via $\tilde{T}(m) = T(\operatorname{tr} m)$. This, in turn, induces a well-defined trace that can be applied to any endomorphism of a finitely generated free left R-module.

If P is a finitely generated projective R-module, it is a summand of a finitely generated free module, R^n, for some n. The projection $R^n \to P$ can be regarded as an idempotent in $M_n(R)$. Stallings shows that in this way, T induces a well-defined homomorphism $\tilde{T}$: $K_0(R) \to$ abelian group of values of T.

As an example, define $t_G: F[G] \to F$ by $t_G(\sum_{g\in G} a_g g) = a_1$. t_G is a trace on $F[G]$.

LEMMA 3. *Let R be a ring of* ch $p > 0$ *and suppose T is a trace function on R. If $m \in M_n(R)$ then $\tilde{T}(m^p) = T((\operatorname{tr} m)^p)$.*

Proof. Write $m = \sum r_{ij} e_{ij}$, where $r_{ij} \in R$ and e_{ij} are the usual matrix units.

$$m^p = \sum r_{ii}^p e_{ii} + \text{sum of commutators}.$$

So, $\tilde{T}(m^p) = T(\sum r_{ii}^p)$. But $(\sum_i r_{ii})^p = \sum r_{ii}^p +$ sum of commutators. Consequently, $T(\sum r_{ii}^p) = T((\sum r_{ii})^p)$. Therefore, $\tilde{T}(m^p) = T((\operatorname{tr} m)^p)$. ∎

Let G be a torsion free noetherian group. In this section we record a slight generalization of Formanek's theorem about idempotents in $F[G]$ [6]. Section 3 is best read side-by-side with his paper. For a fixed $1 \neq x \in G$, Formanek defines a complicated sequence of traces, T_i: $k[G] \to k$ when ch $k = p > 0$.

LEMMA 4. *If $m \in M_n(k[G])$ then $\tilde{T}_i(m^p) = [\tilde{T}_{i+1}(m)]^p$.*

Proof. Lemma 6 of [6] proves our lemma for $m \in R$. By our Lemma 3,

$$\begin{aligned}\tilde{T}_i(m^p) &= T_i((\operatorname{tr} m)^p), \text{ so we can now conclude}\\ &= [T_{i+1}((\operatorname{tr} m))]^p\\ &= [\tilde{T}_{i+1}(m)]^p. \quad ∎\end{aligned}$$

Now one can repeat Formanek's argument verbatim to prove the following technical result. Suppose G is torsion free noetherian and $1 \neq x \in G$. Let F be a field of arbitrary characteristic. If $e \in M_n(F[G])$ is idempotent, define $x(e) = \sum_{g \text{ conjugate to } x} t_G(\operatorname{tr} e) g^{-1})$. Then $x(e) = 0$.

Let i: $Z \to F$ denote the uniquely determined ring homomorphism.

THEOREM 4. *Let G be a torsion free noetherian group. Then $\tilde{t}_G = i \circ \pi$ as functions from $K_0(F[G])$ into F.*

Proof. Let $e \in M_n(F[G])$ be an idempotent.

If $\epsilon: F[G] \to F$ is the augmentation homomorphism then $\epsilon(e)$ is an idempotent in $M_n(F)$. It is easy to see that when e corresponds to the projective $F[G]$-module P then $\epsilon(e)$ corresponds to the F-vector space $F \otimes_{F[G]} P$. Therefore $\mathrm{tr}(\epsilon(e)) = i(\dim_F(F \otimes_{F[G]} P))$.

On the other hand, $\mathrm{tr}(\epsilon(e)) = \sum_{g \in G} t_G((\mathrm{tr}\, e)\, g^{-1}) = t_G(\mathrm{tr}\, e) + \sum_x x(e)$, where x ranges over each nonidentity conjugacy class of G. By the preceding paragraph, $\mathrm{tr}(\epsilon(e)) = t_G(\mathrm{tr}\, e)$. ∎

When F has characteristic 0, Theorem 4 shows that $\tilde{t}_G$ is a K_0-character which is 1 on $F[G]$. Consequently, a new proof of Theorem 3 follows from

LEMMA 5. *Let G be an arbitrary group with a normal subgroup H of finite index. Then $| G : H |\, \tilde{t}_G = \tilde{t}_H \circ (\ )_H$ as functions on $K_0(F[G])$.*

Proof. Suppose $e_1, \ldots, e_n$ is the canonical $F[G]$-basis for $F[G]^n$. With respect to this basis, $\alpha \in \mathrm{End}_{F[G]}(F[G]^n)$ can be represented by a matrix $(a_{ij}) \in M_n(F[G])$.

If we forget some of the module structure on $F[G]$ then α is "also" an $F[H]$-endomorphism of $F[G]^n$; we write $\alpha_H \in \mathrm{End}_{F[H]}(F[G]^n)$. If $1 = g_1, g_2, \ldots, g_s$ is a right transversal to H in G then $\{g_i e_j \mid 1 \leqslant i \leqslant s, 1 \leqslant j \leqslant n\}$ is an $F[H]$-basis for $F[G]^n$. With respect to this basis, we determine the matrix representing α_H.

$$\begin{aligned}
\alpha_H(g_i e_j) &= \alpha(g_i e_j) \\
&= g_i \alpha(e_j) \\
&= g_i \left(\sum_l a_{lj} e_l \right) \\
&= \sum_l (g_i a_{lj} g_i^{-1}) g_i e_l .
\end{aligned}$$

Thus the contribution of the "$g_i e_j$" column of the matrix to $\tilde{t}_H(\alpha_H)$ is t_H of the projection of $g_i a_{jj} g_i^{-1}$ into $F[H]$. That is the same as

$$t_G(g_i a_{jj} g_i^{-1}) = t_G(a_{jj}).$$

Consequently,

$$\begin{aligned}
\tilde{t}_H(\alpha_H) &= \sum_j \sum_i t_G(a_{jj}) \\
&= | G : H |\, t_G(\mathrm{tr}\, \alpha) \\
&= | G : H |\, \tilde{t}_G(\alpha). \quad ∎
\end{aligned}$$

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